

# Critical behavior of Binder ratios and ratios of higher order cumulants of conserved charges in QCD deconfinement phase transition

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Binder liked ratios of baryon number are firstly suggested in relativistic heavy ion collisions. Using 3D-Ising model, the critical behavior of Binder ratios and ratios of higher order cumulants of order parameter are fully presented. Binder ratio is shown to be a step function of temperature. The critical point is the intersection of the ratios of different system sizes between two platforms. From low to high temperature through the critical point, the ratios of third order cumulants change their values from negative to positive in a valley shape, and ratios of fourth order cumulants oscillate around zero. The normalized ratios, like the Skewness and Kurtosis, do not diverge with correlation length, in contrary with corresponding cumulants. Applications of these characters in search critical point in relativistic heavy ion collisions are discussed.

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One of the main goals of current relativistic heavy ion experiments is to locate the critical point of QCD deconfinement phase transition. The critical character is that the correlation length  $\xi$  goes to infinite larger at infinite system. For finite system, like the formed one in relativistic heavy ion collisions, the correlation length should be a finite maximum. Therefore, the various correlation length related observables are suggested in relativistic heavy ion collisions [1].

It has been recently shown that near the critical point, the density-density correlator of baryon-number follow the same power law behavior as the correlator of the sigma field, which is associated with the chiral order parameter [2, 3]. Therefore, the baryon number is considered as an equivalent order parameter of formed system in nuclear collisions [4].

From statistic physics, it also shows that the susceptibilities of order parameter is directly related to the fluctuations of conserved charges, i.e.,

$$\langle \delta N^i \rangle = VT\chi_i. \quad (1)$$

$\chi_i$  is the  $i$ th order susceptibility.  $\langle \delta N^i \rangle = \langle (N - \bar{N})^i \rangle$  is the  $i$ th order cumulants of the conserved charge number  $N$ . For three flavor QCD, the conserved charges are baryon-number, strangeness, and electric charge [5].

The third and forth order cumulants of conserved charges are defined respectively as,

$$K_3 = \langle \delta N^3 \rangle, \quad K_4 = \langle \delta N^4 \rangle - 3\langle \delta N^2 \rangle^2. \quad (2)$$

In the vicinity of critical point, they are argued to be proportional to the higher power of correlation length, i.e.,  $\xi^{4.5}$  and  $\xi^7$  [6, 7], respectively. So they are more sensitive to the correlation length, and highly recommended.

In experiments [8], properly normalized cumulants, i.e., Skewness and Kurtosis,

$$K_3/K_2^{3/2} = \frac{\langle \delta N^3 \rangle}{\langle \delta N^2 \rangle^{3/2}}, \quad K_4/K_2^2 = \frac{\langle \delta N^4 \rangle}{\langle \delta N^2 \rangle^2} - 3, \quad (3)$$

are actually presented. As the second cumulant is also proportional to a certain power of correlation length [9], if such normalized Skewness and Kurtosis diverge with correlation length is not clear from the theoretical point of view.

From theoretical side, the ratios of high order cumulants to the second one, e.g.,

$$R_{3,2} = \frac{\langle \delta N^3 \rangle}{\langle \delta N^2 \rangle}, \quad R_{4,2} = \frac{\langle \delta N^4 \rangle}{\langle \delta N^2 \rangle} - 3\langle \delta N^2 \rangle. \quad (4)$$

are estimated [10–14]. The Lattice QCD with two light quark degrees of freedom shows that these ratios of the baryon number, strangeness, and electric charge have pronounced peaks from low to high temperature in the transition region of chiral symmetry break [10]. The effective models in the mean-field approximation also shows that there are peak, valley, and oscillating structures near the deconfinement and chiral phase transitions [11, 12, 14]. However, all these are obtained under some approximations due to the difficulties in Lattice QCD calculations [15] and model estimations [13].

Although the concrete form of interactions varies from one system to another, according to the theory of universality, the critical exponent of equivalent measurement is identical in the same universality class. This allows us to study the critical behavior of complex system by known simple one.

It is known that the QCD deconfinement phase transition corresponds to the restoration of  $O(1)$  symmetry, which is the same universality class of 3D-Ising model [16]. Therefore, the critical behavior of all above mentioned cumulants of baryon number can be easily obtained from the corresponding cumulants of order parameter in 3D-Ising model.

Moreover, it is known in statistical physics that the Binder ratio of order parameter is a direct location of critical point [17]. Generally, the Binder liked ratios are

normalized raw moments of order parameter. The third and fourth Binder liked cumulant ratios can be simply defined as,

$$B_3 = \frac{\langle M^3 \rangle}{\langle M^2 \rangle^{3/2}}, \quad B_4 = \frac{\langle M^4 \rangle}{\langle M^2 \rangle^2}. \quad (5)$$

Here we take 3D-Ising model as an example. The order parameter in the model is the magnet  $M = \sum_{i=1}^{N_L} \vec{s}_i / N_L$  of spin  $\vec{s}$  in all lattice sites  $N_L$ .

Equivalently, the order parameter in relativistic heavy ion collisions is the baryon number. The temperature, or the controlling parameter, is the incident energy. The size of the formed system is mainly determined by the overlapped area, i.e., centralities. So if we pass through the region of critical incident energies in relativistic heavy ion collisions, the Binder ratios of baryon number can be served as a good location of critical point of QCD deconfinement phase transition.

In this paper, we firstly present the critical behavior of Binder ratios in 3D-Ising model, and demonstrate why they are helpful, in particular, in locating the critical point in relativistic heavy ion collisions. Then, the critical behavior of Skewness, Kurtosis,  $R_{3,2}$ , and  $R_{4,2}$  are presented and discussed, respectively. Meanwhile, from finite-size scaling of the susceptibilities, the critical behavior of those ratios are estimated model independently. Finally, the conclusions are drawn.

The critical behavior of Binder ratios,  $B_3$  and  $B_4$ , in 3D-Ising for 4 different lattice sizes are presented in Fig. 1(a) and (b), respectively. Where the simulation of 3D-Ising model is based on the wolff algorithm [18]. We can see that both  $B_3$  and  $B_4$  show a step jump in the vicinity of critical temperature. The physical meaning of this jump is clear.

When the temperature is much lower than the critical one, the system is almost order and the fluctuation of order parameter is very small, i.e.,

$$\langle M^n \rangle \sim \langle M \rangle^n \quad (\text{for } n = 2, 3, 4, \dots). \quad (6)$$

So it results the lower platform, which is 1 for all orders of Binder ratios at all system sizes, as shown in Fig. 1. When the temperature approaches to critical one, the correlation length starts to increase with temperature and the fluctuations become larger and larger. Their critical behavior is system size dependent and described by finite-size scaling. Only at critical temperature, all size curves intersect to the *fixed point*, where they are system size independent [19], as shown in Fig. 1. When the temperature is much higher than the critical one, the system is totally disordered. It approaches again to a constant. This forms the platform at high temperature. It is 1.6 and 3 times larger than the lower platforms for the third and fourth order Binder ratios, respectively. So the higher the order of Binder ratio, the larger the gap of the step function.

This step function liked behavior can be served as a very good probe of critical point in relativistic heavy ion

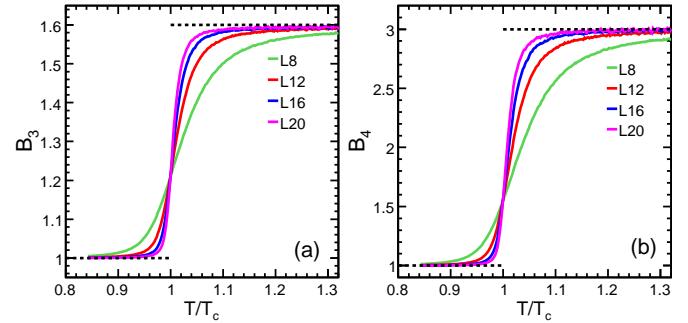


FIG. 1: (Color online) The temperature dependence of Binder ratios in Eq. (5) in the vicinity of critical temperature in 3D-Ising model for 4 different lattice sizes.

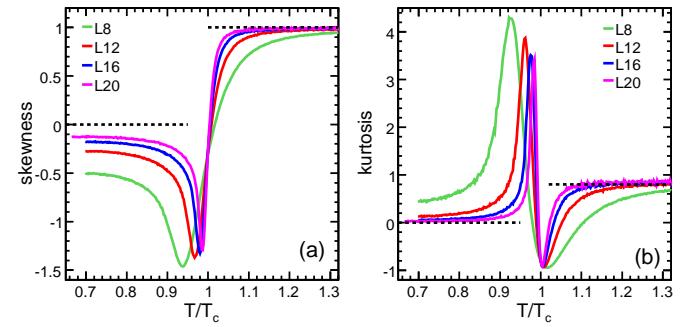


FIG. 2: (Color online) The temperature dependence of Kurtosis (a) and Skewness (b) in Eq. (3) in the vicinity of critical temperature in 3D Ising model for 4 different lattice sizes.

collisions, where critical incident energy is difficult to assign precisely in priori. So if we scan incident energies, and observe two platforms at low and high energy regions, respectively, then the critical one is most probably between them. We can finely tune the incident energy in the region and precisely determine the critical energy and exponents.

The Skewness and Kurtosis of order parameter in 3D-Ising model for 4 different lattice sizes are presented in Fig. 2(a) and (b), respectively. We can see from the figure that they change sharply in the vicinity of the critical temperature. The Skewness first drops down and then goes up, and Kurtosis oscillates with temperature. Their values are system size dependent. Their signs change respectively near the critical point. Former in Fig. 2(a) changes from negative to positive when the temperature is increased through the critical point, while the later in Fig. 2(b) becomes negative only when the temperature is close to the critical point. The sign change in Skewness, or third order cumulants, is expected in effective models [12, 14, 20].

As we know that the Skewness and Kurtosis measure the symmetry and sharpness of the distribution, respectively. The distributions of order parameter  $M$  near the critical point at system size  $L = 8$  are shown in

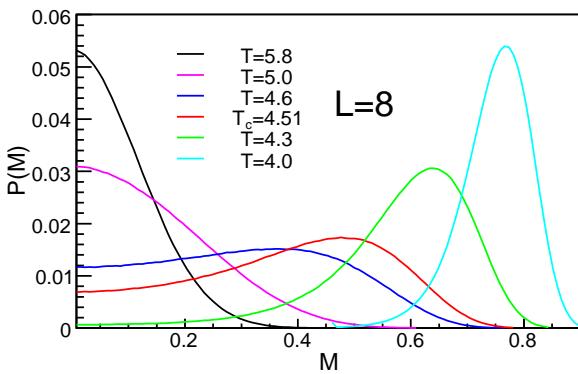


FIG. 3: (Color online) The distributions of order parameter near critical temperatures in 3D-Ising model at system size  $L = 8$ .

Fig. 3. Where we can clearly see that the long tail of the distributions changes from the left to the right side when the temperature is increased through the critical point, and the peak of the distribution vary from sharp to flat when temperature is approached the critical point. The same trend has been observed in percolation model, in studying clusterization phenomena in nuclear multi-fragmentation [21].

This character can also be served as a signal associated with the appearance of critical point in relativistic heavy ion collisions. If we observe sign change of Skewness (Kurtosis) of baryon number at a certain incident energy region, it most probably predicts the appearance of critical point in the nearby incident energy region.

The Skewness and Kurtosis also converge to two constants when the temperature is away from critical point, as shown in Fig. 2(a) and (b). But the constants at low and high temperatures are close to zero and 1, respectively. The gap between them are small and does not change very much with the order of cumulants, unlike the Binder ratio.

Moreover, all size curves of Skewness (Kurtosis) intersect at critical point. This can be easily understood from finite-size scaling of susceptibilities, i.e.,

$$\chi_i(t, L) = L^{\gamma_i/\nu} P_{\chi_i}(tL^{1/\nu}). \quad (7)$$

Where the  $\gamma_i$  is the critical exponents of  $i$ th order susceptibility, and  $\nu$  is the critical exponent of correlation length  $\xi_\infty = t^{-\nu}$  at infinite system.  $t = \frac{T-T_c}{T_c}$  is reduced temperature, and  $T_c$  is critical temperature. In the vicinity of critical point, the correlation length at finite system is approximately the same order of the system size, i.e.,  $\xi \sim L = V^{1/3}$ .

For  $\chi_3$  and  $\chi_4$ ,  $\gamma_3/\nu = 4.5$ ,  $\gamma_4/\nu = 7$ , respectively [6]. So the critical behavior of the Skewness and Kurtosis in

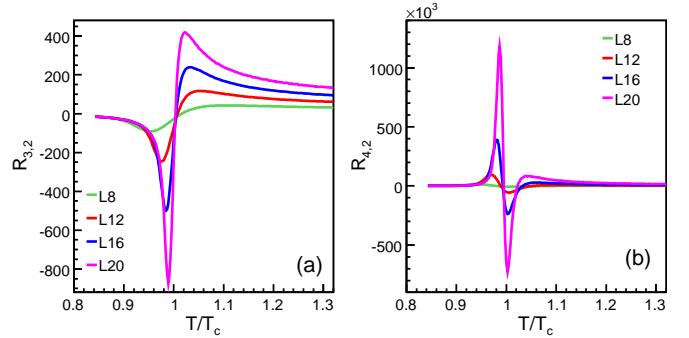


FIG. 4: (Color online) The temperature dependence of  $R_{3,2}$  (a), and  $R_{4,2}$  (b) in the vicinity of critical temperature in 3D-Ising model for 4 different lattice sizes.

Eq. (3) are,

$$\begin{aligned} K_3/K_2^{3/2} &= \frac{VT\chi_3}{(VT)^{3/2}\chi_2^{3/2}} \sim \frac{L^3 L^{4.5} P_{\chi_3}(tL^{1/\nu})}{L^{4.5} L^3 T^{1/2} P_{\chi_2}^{3/2}(tL^{1/\nu})} \\ &= T^{-1/2} F_S(tL^{1/\nu}), \\ K_4/K_2^2 &= \frac{VT\chi_4}{(VT)^2\chi_2^2} \sim \frac{L^3 L^7 P_{\chi_4}(tL^{1/\nu})}{L^6 L^4 T P_{\chi_2}^2} \\ &= T^{-1} F_K(tL^{1/\nu}). \end{aligned} \quad (8)$$

They no long diverge with correlation length, or system size. At the critical temperature  $t = 0$ , the scaling function, i.e.,  $F_S(0)$  or  $F_K(0)$ , is system size independent constant. All size curves intersect to the constant, i.e., the *fixed point* [19].

From this simple estimation and Fig. 2, we can see that normalized high order cumulants, i.e., Skewness and Kurtosis, do not directly diverge with correlation length any more, different from corresponding cumulants,  $K_3$  and  $K_4$ , which are proportional to  $\xi^{4.5}$  and  $\xi^7$ , respectively [6, 7].

The  $R_{3,2}$ , and  $R_{4,2}$  of order parameter in 3D-Ising model for 4 different lattice sizes are presented in Fig. 4(a) and (b), respectively. We can see again from Fig. 4(a) that  $R_{3,2}$  changes its value sharply from negative to positive when temperature is increased through the critical point.  $R_{4,2}$  in Fig. 4(b) oscillates greatly with temperature near the critical point. These qualitative features, i.e., sign change in third moment, and oscillating structure in forth cumulants, are consistent with estimations of effective models [12, 14, 20].

$R_{3,2}$  and  $R_{4,2}$  are very sensitive to the system size, or correlation length. Their values become very large when system size increases. The critical exponent of  $R_{3,2}$ , and  $R_{4,2}$  can be roughly estimated from finite-size scaling of

susceptibilities, i.e.,

$$\begin{aligned}
R_{3,2} &= \frac{VT\chi_3}{VT\chi_2} = \frac{L^3\xi^{4.5}P_{\chi_3}(tL^{1/\nu})}{L^3\xi^2P_{\chi_2}(tL^{1/\nu})} \\
&= \xi^{2.5}F_{R_{3,2}}(tL^{1/\nu}) \\
R_{4,2} &= \frac{VT\chi_4}{VT\chi_2} = \frac{L^3\xi^7P_{\chi_4}(tL^{1/\nu})}{L^3\xi^2P_{\chi_2}(tL^{1/\nu})} \\
&= \xi^5F_{R_{4,2}}(tL^{1/\nu}).
\end{aligned} \tag{9}$$

So  $R_{3,2}$  and  $R_{4,2}$  diverge with correlation length as  $\xi^{2.5}$  and  $\xi^5$ , respectively.

In this paper, the measurements of Binder liked ratios of conserved charges are firstly suggested in relativistic heavy ion collisions. Using 3D-Ising model, it is shown that near the critical temperature, Binder ratios is a step function of temperature. The gap of the step function is 1.6 and 3 times wider for the third and forth order Binder ratios, respectively. This can be served as a good identification of critical behavior in relativistic heavy ion collisions, where the critical incident energy is unknown in prior. The critical point is the intersection of Binder ratios of different size systems between two platforms.

The critical behavior of Skewness, Kurtosis,  $R_{3,2}$  and

$R_{4,2}$  at various system sizes are also studied by 3D-Ising model, and estimated by finite size scaling. When the temperature is increased through the critical point, the ratios of the third order cumulants change their values from negative to positive in a valley shape, and ratios of fourth order cumulants oscillate around zero. All size curves of Skewness (Kurtosis) intersect at the critical point. The normalized ratios, like the Skewness and Kurtosis, do not diverge with correlation length. While, unnormalized ratios,  $R_{3,2}$  and  $R_{4,2}$ , are divergent with correlation length. They are proportional to  $\xi^{2.5}$  and  $\xi^5$ , respectively, and very sensitive to the system size near the critical temperature.

These critical characters may show up at the energy dependence of corresponding ratios of conserved charges. Their behavior at coming relativistic heavy ion experiments at RHIC, SPS, and FAIR are called for.

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